COMMUTATIVE SUBALGEBRAS OF THE ALGEBRA OF SMOOTH OPERATORS

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Abstract. We consider the Fréchet *-algebra $\mathcal{L}(s',s) \subseteq \mathcal{L}(\ell_2)$ of the so-called smooth operators, i.e. continuous linear operators from the dual s' of the space s of rapidly decreasing sequences into s. This algebra is a non-commutative analogue of the algebra s. We characterize all closed commutative *-subalgebras of $\mathcal{L}(s',s)$ which are at the same time isomorphic to closed *-subalgebras of s and we provide an example of a closed commutative *-subalgebra of $\mathcal{L}(s',s)$ which cannot be embedded into s.

1. Introduction

The algebra $\mathcal{L}(s',s)$ can be represented as the algebra

$$\mathcal{K}_\infty := \{(x_{j,k})_{j,k\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}^2} : \sup_{j,k\in\mathbb{N}} |x_{j,k}| j^q k^q < \infty \text{ for all } q \in \mathbb{N}_0\}$$

of rapidly decreasing matrices (with matrix multiplication and matrix complex conjugation). Another representation of $\mathcal{L}(s',s)$ is the algebra $\mathcal{S}(\mathbb{R}^2)$ of Schwartz functions on $\mathbb{R}^2$ with the Volterra convolution

$$(f\cdot g)(x,y) := \int_{\mathbb{R}} f(x,z)g(z,y)\mathrm{d}z$$

as multiplication and the involution

$$f^*(x,y) := \overline{f(y,x)}.$$

In these forms, the algebra $\mathcal{L}(s',s)$ usually appears and plays a significant role in K-theory of Fréchet algebras (see Bhatt and Inoue [1, Ex. 2.12], Cuntz [5, p. 144], [6, p. 64–65], Glöckner and Langkamp [10], Phillips [13, Def. 2.1]) and in C^*-dynamical systems (Elliot, Natsume and Nest [8, Ex. 2.6]). Very recently, Piszczek obtained several rusults concerning closed ideals, automatic continuity (for positive functionals and derivations), amenability and Jordan decomposition in $\mathcal{K}_\infty$ (see Piszczek [16, 15] and his forthcoming papers 'Automatic continuity and amenability in the non-commutative Schwartz space' and 'The noncommutative Schwartz space is weakly amenable'). Moreover, in the context of algebras of unbounded operators, the algebra $\mathcal{L}(s',s)$ appears in the book [17] as

$$\mathbb{B}_1(s) := \{x \in \mathcal{L}(\ell_2)\colon x\ell_2 \subseteq s, x^*\ell_2 \subseteq s \text{ and } \overline{axb} \text{ is nuclear for all } a,b \in \mathcal{L}^*(s)\},$$

where $\mathcal{L}^*(s)$ is the so-called maximal O^*-algebra on s (see also [17, Def. 2.1.6, Prop. 2.1.8, Def. 5.1.3, Cor. 5.1.18, Prop. 5.4.1 and Prop. 6.1.5]).

The algebra of smooth operators can be seen as a noncommutative analogue of the commutative algebra s. The most important features of this algebra are the following:

- it is isomorphic as a Fréchet space to the Schwartz space $\mathcal{S}(\mathbb{R})$ of smooth rapidly decreasing functions on the real line;

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- it has several representations as algebras of operators acting between natural spaces of distributions and functions (see [7, Th. 1.1]);
- it is a dense *-subalgebra of the C^*-algebra $\mathcal{K}(\ell_2)$ of compact operators on ℓ_2;
- it is (properly) contained in the intersection of all Schatten classes $\mathcal{S}_p(\ell_2)$ over $p>0$; in particular $\mathcal{L}(s',s)$ is contained the class $\mathcal{HS}(\ell_2)$ of Hilbert-Schmidt operators, and thus it is a unitary space;
- the operator C^*-norm $||\cdot||_{\ell_2\to\ell_2}$ is the so-called dominating norm on that algebra (the dominating norm property is a key notion in the structure theory of nulcear Fréchet spaces – see [3, Prop. 3.2] and [12, Prop. 31.5]).

The main result of the present paper is a characterization of closed *-subalgebras of $\mathcal{L}(s',s)$ which are at the same time isomorphic as Fréchet *-algebras to closed *-subalgebras of s (Theorem 6.2). It turns out that these are exactly those subalgebras which satisfy the classical condition (Ω) of Vogt. Then in Theorem 6.10 we give an example of a closed commutative *-subalgebra of $\mathcal{L}(s',s)$ which does not satisfy this condition.

In order to prove this result we characterize in Section 4 closed *-subalgebras of Köthe sequence algebras (Proposition 4.3). In particular, we give such a description for closed *-subalgebras of s (Corollary 4.4). In Section 5 we describe all closed *-subalgebras of $\mathcal{L}(s',s)$ as suitable Köthe sequence algebras (see Corollary 5.4 and compare with [3, Th. 4.8])

The present paper is a continuation of [3] and [7] and it focuses on description of closed commutative *-subalgebras of $\mathcal{L}(s',s)$ (especially those with the property (Ω)). Most of the results have been already presented in the author PhD dissertation [2].

2. Notation and terminology

Throughout the paper, $\mathbb{N}$ will denote the set of natural numbers $\{1,2,\ldots\}$ and $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$.

By a *projection* on the complex separable Hilbert space ℓ_2 we always mean a continuous orthogonal (i.e. self-adjoint) projection.

By e_k we denote the vector in $\mathbb{C}^{\mathbb{N}}$ whose k-th coordinate equals 1 and the others equal 0.

By a *Fréchet space* we mean a complete metrizable locally convex space over $\mathbb{C}$ (we will not use locally convex spaces over $\mathbb{R}$). A *Fréchet algebra* is a Fréchet space which is an algebra with continuous multiplication. A *Fréchet *-algebra* is a Fréchet algebra with continuous involution.

For locally convex spaces E,F, we denote by $\mathcal{L}(E,F)$ the space of all continuous linear operators from E to F. To shorten notation, we write $\mathcal{L}(E)$ instead of $\mathcal{L}(E,E)$.

We use the standard notation and terminology. All the notions from functional analysis are explained in [4] or [12] and those from topological algebras in [9] or [20].

3. Preliminaries

The space s and its dual. We recall that the *space of rapidly decreasing sequences* is the Fréchet space

$$s:=\left\{\xi=(\xi_j)_{j\in\mathbb{N}}\in\mathbb{C}^{\mathbb{N}}:|\xi|_q:=\left(\sum_{j=1}^{\infty}|\xi_j|^2j^{2q}\right)^{1/2}<\infty\text{ for all }q\in\mathbb{N}_0\right\}$$

with the topology corresponding to the system $(|\cdot|_q)_{q\in\mathbb{N}_0}$ of norms. We may identify the strong dual of s (i.e. the space of all continuous linear functionals on s with the topology of uniform convergence on bounded subsets of s, see e.g. [12, Def. on p. 267]) with the *space of slowly increasing sequences*

$$s':=\left\{\xi=(\xi_j)_{j\in\mathbb{N}}\in\mathbb{C}^{\mathbb{N}}:|\xi|'_q:=\left(\sum_{j=1}^{\infty}|\xi_j|^2j^{-2q}\right)^{1/2}<\infty\text{ for some }q\in\mathbb{N}_0\right\}$$

equipped with the inductive limit topology given by the system $(|\cdot|'_q)_{q\in\mathbb{N}_0}$ of norms (note that for a fixed q, $|\cdot|'_q$ is defined only on a subspace of s'). More precisely, every $\eta\in s'$ corresponds to the continuous linear functional on s:

$$\xi\mapsto\langle\xi,\eta\rangle:=\sum_{j=1}^{\infty}\xi_j\overline{\eta_j}$$

(note the conjugation on the second variable). These functionals are continuous, because, by the Cauchy-Schwartz inequality, for all $q\in\mathbb{N}_0$, $\xi\in s$ and $\eta\in s'$ we have

$$|\langle\xi,\eta\rangle|\leq|\xi|_q|\eta|'_q.$$

Conversely, one can show that for each continuous linear functional y on s there is $\eta\in s'$ such that $y=\langle\cdot,\eta\rangle$.

Similarly, we identify each $\xi\in s$ with the continuous linear functional on s':

$$\eta\mapsto\langle\eta,\xi\rangle:=\sum_{j=1}^{\infty}\eta_j\overline{\xi_j}.$$

In particular, for each continuous linear functional y on s' there is $\xi\in s$ such that $y=\langle\cdot,\xi\rangle$.

We emphasize that the "scalar product" $\langle\cdot,\cdot\rangle$ is well-defined on $s\times s'\cup s'\times s$ and, of course, on $\ell_2\times\ell_2$.

Property (DN) for the space s . Closed subspaces of the space s can be characterized by the so-called property (DN).

Definition 3.1. A Fréchet space $(X,(||\cdot||_q)_{q\in\mathbb{N}_0})$ has the *property* (DN) (see [12, Def. on p. 359]) if there is a continuous norm $||\cdot||$ on X such that for all $q\in\mathbb{N}_0$ there is $r\in\mathbb{N}_0$ and $C>0$ such that

$$||x||_q^2\leq C||x||\,||x||_r$$

for all $x\in X$. The norm $||\cdot||$ is called a *dominating norm*.

Vogt (see [19] and [12, Ch. 31]) proved that a Fréchet space is isomorphic to a closed subspace of s if and only if it is nuclear and it has the property (DN).

The (DN) condition for the space s reads as follows (see [12, Lemma 29.2(3)] and its proof).

Proposition 3.2. *For every $p\in\mathbb{N}_0$ and $\xi\in s$ we have*

$$|\xi|_p^2\leq||\xi||_{\ell_2}|\xi|_{2p}.$$

In particular, the norm $||\cdot||_{\ell_2}$ is a dominating norm on s.

The algebra $\mathcal{L}(s',s)$. It is a simple matter to show that $\mathcal{L}(s',s)$ with the topology of uniform convergence on bounded sets in s' is a Fréchet space. It is isomorphic to $s\widehat{\otimes}s$, the completed tensor product of s (see [11, §41.7 (5)] and note that, s being nuclear, there is only one tensor topology), and thus $\mathcal{L}(s',s)\cong s$ as Fréchet spaces (see e.g. [12, Lemma 31.1]). Moreover, it is easily seen that $(||\cdot||_q)_{q\in\mathbb{N}_0}$,

$$||x||_q:=\sup_{|\xi|'_q\leq 1}|x\xi|_q,$$

is a fundamental sequence of norms on $\mathcal{L}(s',s)$.

Let us introduce multiplication and involution on $\mathcal{L}(s',s)$. First observe that s is a dense subspace of ℓ_2, ℓ_2 is a dense subspace of s', and, moreover, the embedding maps $j_1\colon s\hookrightarrow\ell_2$, $j_2\colon\ell_2\hookrightarrow s'$ are continuous. Hence,

$$\iota\colon\mathcal{L}(s',s)\hookrightarrow\mathcal{L}(\ell_2),\quad\iota(x):=j_1\circ x\circ j_2,$$

is a well-defined (continuous) embedding of $\mathcal{L}(s',s)$ into the C^*-algebra $\mathcal{L}(\ell_2)$, and thus it is natural to define a multiplication on $\mathcal{L}(s',s)$ by

$$xy:=\iota^{-1}(\iota(x)\circ\iota(y)),$$

i.e.

$$xy = x \circ j \circ y,$$

where $j := j_2 \circ j_1 \colon s \hookrightarrow s'$. Similarly, an involution on $\mathcal{L}(s', s)$ is defined by

$$x^* := \iota^{-1}(\iota(x)^*),$$

where $\iota(x)^*$ is the hermitian adjoint of $\iota(x)$. One can show that these definitions are correct, i.e. $\iota(x) \circ \iota(y), \iota(x)^* \in \iota(\mathcal{L}(s', s))$ for all $x, y \in \mathcal{L}(s', s)$ (see also [3, p. 148]).

From now on, we will identify $x \in \mathcal{L}(s', s)$ and $\iota(x) \in \mathcal{L}(\ell_2)$ (we omit ι in the notation).

A Fréchet algebra E is called *locally m-convex* if E has a fundamental system of submultiplicative seminorms. It is well-known that $\mathcal{L}(s', s)$ is locally m-convex (see e.g. [13, Lemma 2.2]), and moreover, the norms $||\cdot||_q$ are submultiplicative (see [3, Prop. 2.5]). This shows simultaneously that the multiplication introduced above is separately continuous, and thus, by [20, Th. 1.5], it is jointly continuous. Moreover, by [9, Cor. 16.7], the involution on $\mathcal{L}(s', s)$ is continuous.

We may summarize this paragraph by saying that $\mathcal{L}(s', s)$ is a noncommutative *-subalgebra of the C^*-algebra $\mathcal{L}(\ell_2)$ which is (with its natural topology) a locally m-convex Fréchet *-algebra isomorphic as a Fréchet space to s.

4. Köthe algebras

In this section we collect and prove some results on Köthe algebras which are known for specialists but probably never published.

Definition 4.1. A matrix $A = (a_{j,q})_{j \in \mathbb{N}, q \in \mathbb{N}_0}$ of non-negative numbers such that

(i) for each $j \in \mathbb{N}$ there is $q \in \mathbb{N}_0$ such that $a_{j,q} > 0$

(ii) $a_{j,q} \leq a_{j,q+1}$ for $j \in \mathbb{N}$ and $q \in \mathbb{N}_0$

is called a *Köthe matrix*.

For $1 \leq p < \infty$ and a Köthe matrix A we define the *Köthe space*

$$\lambda^p(A) := \left\{\xi = (\xi_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : |\xi|_{p,q} := \left(\sum_{j=1}^{\infty} |\xi_j|^p a_{j,q}|^p\right)^{1/p} < \infty \text{ for all } q \in \mathbb{N}_0\right\}$$

and for $p = \infty$

$$\lambda^\infty(A) := \left\{\xi = (\xi_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : |\xi|_{\infty,q} := \sup_{j\in\mathbb{N}} |\xi_j| a_{j,q} < \infty \text{ for all } q \in \mathbb{N}_0\right\}$$

with the locally convex topology given by the seminorms $(|\cdot|_{p,q})_{q\in\mathbb{N}_0}$ (see e.g. [12, Def. p. 326]).

Sometimes, for simplicity, we will write $\lambda^\infty(a_{j,q})$ (i.e. only the entries of the matix) instead of $\lambda^\infty(A)$.

It is well-known (see [12, Lemma 27.1]) that the spaces $\lambda^p(A)$ are Fréchet spaces and sometimes they are Fréchet *-algebras with pointwise multiplication and conjugation (e.g. if $a_{j,q} \geq 1$ for all $j \in \mathbb{N}$ and $q \in \mathbb{N}_0$, see also [14, Prop. 3.1]); in that case they are called *Köthe algebras*.

Clearly, s is the Köthe space $\lambda^2(A)$ for $A = (j^q)_{j\in\mathbb{N}, q\in\mathbb{N}_0}$ and it is a Fréchet *-algebra. Moreover, since the matrix A satisfies the so-called Grothendieck-Pietsch condition (see e.g. [12, Prop. 28.16(6)]), s is nuclear, and thus it has also other Köthe space representations (see again [12, Prop. 28.16 & Ex. 29.4(1)]), i.e. for all $1 \leq p \leq \infty$, $s = \lambda^p(A)$ as Fréchet spaces.

We use ℓ_2-norms in the definition of s to clarify our ideas, for example we have $|\xi|_0 = ||\xi||_{\ell_2}$ for $\xi \in s$ and $|\eta|_0' = ||\eta||_{\ell_2}$ for $\eta \in \ell_2$. However, in some situations the supremum norms $|\cdot|_{\infty,q}$ (as they are relatively easy to compute) or the ℓ_1-norms will be more convenient.

Proposition 4.2. *Let $A = (a_{j,q})_{j\in\mathbb{N}, q\in\mathbb{N}_0}$, $B = (b_{j,q})_{j\in\mathbb{N}, q\in\mathbb{N}_0}$ be Köthe matrices and for a bijection $\sigma \colon \mathbb{N} \to \mathbb{N}$ let $A_\sigma := (a_{\sigma(j),q})_{j\in\mathbb{N}, q\in\mathbb{N}_0}$. Assume that $\lambda^\infty(A)$ and $\lambda^\infty(B)$ are Fréchet *-algebras. Then the following assertions are equivalent:*

(i) $\lambda^\infty(A) \cong \lambda^\infty(B)$ *as Fréchet* **-algebras;*
(ii) *there is a bijection* $\sigma\colon \mathbb{N} \to \mathbb{N}$ *such that* $\lambda^\infty(A_\sigma) = \lambda^\infty(B)$ *as Fréchet* **-algebras;*
(iii) *there is a bijection* $\sigma\colon \mathbb{N} \to \mathbb{N}$ *such that* $\lambda^\infty(A_\sigma) = \lambda^\infty(B)$ *as sets;*
(iv) *there is a bijection* $\sigma\colon \mathbb{N} \to \mathbb{N}$ *such that*
(α) $\forall q \in \mathbb{N}_0\ \exists r \in \mathbb{N}_0\ \exists C > 0\ \forall j \in \mathbb{N} \quad a_{\sigma(j),q} \le C b_{j,r}$,
(β) $\forall r' \in \mathbb{N}_0\ \exists q' \in \mathbb{N}_0\ \exists C' > 0\ \forall j \in \mathbb{N} \quad b_{j,r'} \le C' a_{\sigma(j),q'}$.

Proof. (i)$\Rightarrow$(ii) Assume that there is an isomorphism $\Phi\colon \lambda^\infty(A) \to \lambda^\infty(B)$ of Fréchet *-algebras. Clearly, if $\xi^2 = \xi$, then $\Phi(\xi) = \Phi(\xi^2) = (\Phi(\xi))^2$, and the same is true for Φ^{-1}, i.e. Φ maps the idempotents of $\lambda^\infty(A)$ onto the idempotents of $\lambda^\infty(B)$. Hence for a fixed $k \in \mathbb{N}$, there is $I \subset \mathbb{N}$ such that

$$\Phi(e_k) = e_I,$$

where e_I is a sequence which has 1 on an index set $I \subset \mathbb{N}$ and 0 otherwise. Suppose that $|I| \ge 2$ and let $j \in I$. Then $e_I = e_j + e_{I\setminus\{j\}}$, where $e_j \in \lambda^\infty(B)$ and $e_{I\setminus\{j\}} = e_I - e_j \in \lambda^\infty(B)$. Therefore, there are nonempty subsets $I_j, I'_j \subset \mathbb{N}$ such that $\Phi(e_{I_j}) = e_j$ and $\Phi(e_{I'_j}) = e_{I\setminus\{j\}}$. We have

$$e_{I_j} e_{I'_j} = \Phi^{-1}(e_j)\Phi^{-1}(e_{I\setminus\{j\}}) = \Phi^{-1}(e_j e_{I\setminus\{j\}}) = 0,$$

and thus $I_j \cap I'_j = \emptyset$. Consequently,

$$\Phi(e_k) = e_j + e_{I\setminus\{j\}} = \Phi(e_{I_j}) + \Phi(e_{I'_j}) = \Phi(e_{I_j \cup I'_j}),$$

whence $1 = |\{k\}| = |I_j \cup I'_j| \ge 2$, a contradiction. Hence $\Phi(e_k) = e_{n_k}$ for some $n_k \in \mathbb{N}$, i.e. for the bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$ defined by $n_{\sigma(k)} := k$ we have $\Phi(e_{\sigma(k)}) = e_k$. Therefore, a Fréchet *-isomorphism Φ is given by $(\xi_{\sigma(k)})_{k\in\mathbb{N}} \mapsto (\xi_k)_{k\in\mathbb{N}}$ for $(\xi_{\sigma(k)})_{k\in\mathbb{N}} \in \lambda^\infty(A)$, and thus $\lambda^\infty(A_\sigma) = \lambda^\infty(B)$ as Fréchet *-algebras.

(ii)$\Rightarrow$(iii) Obvious.

(iii)$\Rightarrow$(iv) The proof follows from the observation that the identity map $\mathrm{Id}\colon \lambda^\infty(A_\sigma) \to \lambda^\infty(B)$ is continuous (use the closed graph theorem).

(iv)$\Rightarrow$(i) It is easy to see that $\Phi\colon \lambda^\infty(A) \to \lambda^\infty(B)$ defined by $e_{\sigma(k)} \mapsto e_k$ is an isomorphism of Fréchet *-algebras. $\square$

In the following proposition we characterize infinite-dimensional closed *-subalgebras of nuclear Köthe algebras whose elements tends to zero (note that if a Köthe space is contained in ℓ_∞ then it is a Köthe algebra). Consequently, we obtain a characterization of closed *-subalgebras of s (Corollary 4.4).

Proposition 4.3. *For* $\mathcal{N} \subset \mathbb{N}$ *let* $e_\mathcal{N}$ *denote a sequence which has 1 on* $\mathcal{N}$ *and 0 otherwise. Let* $A = (a_{j,q})_{j\in\mathbb{N},q\in\mathbb{N}_0}$ *be a Köthe matrix such that* $\lambda^\infty(A)$ *is nuclear and* $\lambda^\infty(A) \subset c_0$*. Let* E *be an infinite-dimensional closed* **-subalgebra of* $\lambda^\infty(A)$*. Then*

(i) *there is a family* $\{\mathcal{N}_k\}_{k\in\mathbb{N}}$ *of finite nonempty pairwise disjoint sets of natural numbers such that* $(e_{\mathcal{N}_k})_{k\in\mathbb{N}}$ *is a Schauder basis of* E*;*
(ii) $E \cong \lambda^\infty\left(\max_{j\in\mathcal{N}_k} a_{j,q}\right)$ *as Fréchet* **-algebras and the isomorphism is given by* $e_{\mathcal{N}_k} \mapsto e_k$ *for* $k \in \mathbb{N}$*.*

Conversely, if $\{\mathcal{N}_k\}_{k\in\mathbb{N}}$ *is a family of finite nonempty pairwise disjoint sets of natural numbers and* F *is the closed* **-subalgebra of* $\lambda^\infty(A)$ *generated by the set* $\{e_{\mathcal{N}_k}\}_{k\in\mathbb{N}}$*, then*

(iii) $(e_{\mathcal{N}_k})_{k\in\mathbb{N}}$ *is a Schauder basis of* F*;*
(iv) $F \cong \lambda^\infty(\max_{j\in\mathcal{N}_k} a_{j,q})$ *as Fréchet* **-algebras and the isomorphism is given by* $e_{\mathcal{N}_k} \mapsto e_k$ *for* $k \in \mathbb{N}$*.*

Proof. In order to prove (i) and (ii) set

$$\mathcal{N}_0 := \{j \in \mathbb{N} \colon \xi_j = 0 \quad \text{for all } \xi \in E\}$$

and define an equivalence relation $\sim$ on $\mathbb{N}\setminus\mathcal{N}_0$ by

$$i\sim j \Leftrightarrow \xi_i=\xi_j \text{ for all } \xi\in E.$$

Since E is infinite-dimensional, our relation produces infinitely many equivalence classes $\mathcal{N}_k$, say

$$\mathcal{N}_k := [\min(\mathbb{N}\setminus\mathcal{N}_0\cup\ldots\cup\mathcal{N}_{k-1})]_{/\sim}$$

for $k\in\mathbb{N}$.

Fix $\kappa\in\mathbb{N}$ and take $\xi\in E$ such that $\xi_j\neq 0$ for $j\in\mathcal{N}_\kappa$. Denote $\eta_k:=\xi_j$ if $j\in\mathcal{N}_k$. Let

$$\mathcal{M}_1 := \{j\in\mathbb{N}\colon |\eta_j|=\sup_{i\in\mathbb{N}}|\eta_i|\}.$$

Assume we have already defined $\mathcal{M}_1,\ldots,\mathcal{M}_{l-1}$. If there is $j\in\mathbb{N}\setminus\{\mathcal{M}_1\cup\ldots\cup\mathcal{M}_{l-1}\}$ such that $\eta_j\neq 0$ then we define

$$\mathcal{M}_l := \{j\in\mathbb{N}\colon |\eta_j|=\sup\{|\eta_i|\colon i\in\mathbb{N}\setminus\mathcal{M}_1\cup\ldots\cup\mathcal{M}_{l-1}\}\}.$$

Otherwise, denote $\mathcal{I}:=\{1,\ldots,l-1\}$. If this procedure leads to infinite many sets $\mathcal{M}_l$ then we set $\mathcal{I}:=\mathbb{N}$. It is easily seen that for each $l\in\mathcal{I}$ there is $\mathcal{I}_l\subset\mathbb{N}$ such that $\mathcal{M}_l=\bigcup_{k\in\mathcal{I}_l}\mathcal{N}_k$. By assumption $\xi\in c_0$, hence $(|\eta_k|)_{k\in\mathbb{N}}\in c_0$ as well, and thus each $\mathcal{M}_l$ is a finite nonempty set.

We first show that $e_{\mathcal{M}_l}\in E$ for $l\in\mathcal{I}$. For $l\in\mathcal{I}$ fix $m_l\in\mathcal{M}_l$. If $\mathcal{I}=\{1\}$, then $\xi_j=0$ for $j\notin\mathcal{M}_1$, and $e_{\mathcal{M}_1}=\frac{\xi\overline{\xi}}{|\eta_{m_1}|^2}\in E$. Let us consider the case $|\mathcal{I}|>1$. Since in nuclear Fréchet spaces every basis is absolute (and thus unconditional), we have

$$\sum_{l\in\mathcal{I}}|\eta_l|^2 e_{\mathcal{M}_l}=\sum_{j=1}^{\infty}|\xi_j|^2 e_j=\xi\overline{\xi}\in E,$$

and, consequently,

$$x_n:=\sum_{l\in\mathcal{I}}\left(\frac{|\eta_l|}{|\eta_{m_1}|}\right)^{2n}e_{\mathcal{M}_l}=\left(\frac{\xi\overline{\xi}}{|\eta_{m_1}|^2}\right)^n\in E$$

for all $n\in\mathbb{N}$. Then for q and n we get

$$\begin{aligned}|x_n-e_{\mathcal{M}_1}|_{\infty,q}&=\left|\sum_{l\in\mathcal{I}}^{\infty}\left(\frac{|\eta_l|}{|\eta_{m_1}|}\right)^{2n}e_{\mathcal{M}_l}-e_{\mathcal{M}_1}\right|_{\infty,q}=\left|\sum_{l\in\mathcal{I}\setminus\{1\}}\left(\frac{|\eta_l|}{|\eta_{m_1}|}\right)^{2n}e_{\mathcal{M}_l}\right|_{\infty,q}\\&\le\sum_{l\in\mathcal{I}\setminus\{1\}}\left(\frac{|\eta_l|}{|\eta_{m_1}|}\right)^{2n}|e_{\mathcal{M}_l}|_{\infty,q}\le\frac{1}{|\eta_{m_1}|}\left(\frac{|\eta_{m_2}|}{|\eta_{m_1}|}\right)^{2n-1}\sum_{l\in\mathcal{I}\setminus\{1\}}|\eta_l|\,|e_{\mathcal{M}_l}|_{\infty,q}.\end{aligned}$$

Since $(e_j)_{j\in\mathbb{N}}$ is an absolute basis in $\lambda^\infty(A)$, the above series is convergent. Note also that $|\eta_{m_2}|<|\eta_{m_1}|$. This shows that $x_n\to e_{\mathcal{M}_1}$ in $\lambda^\infty(A)$, and $e_{\mathcal{M}_1}\in E$. Assume that $e_{\mathcal{M}_1},\ldots,e_{\mathcal{M}_{l-1}}\in E$. If $|\mathcal{I}|=l-1$ then we are done. Otherwise, $\eta_{m_l}\neq 0$ and

$$x_n^{(l)}:=\left(\frac{\xi\overline{\xi}-\xi\overline{\xi}\sum_{j=1}^{l-1}e_{\mathcal{M}_j}}{|\eta_{m_l}|^2}\right)^n\in E$$

for $n\in\mathbb{N}$. As above we show that $x_n^{(l)}\to e_{\mathcal{M}_l}$ in $\lambda^\infty(A)$, and thus $e_{\mathcal{M}_l}\in E$. Proceeding by induction, we prove that $e_{\mathcal{M}_l}\in E$ for $l\in\mathcal{I}$.

Now, we shall prove that $(e_{\mathcal{N}_k})_{k\in\mathbb{N}}$ is a Schauder basis of E. Choose $\iota\in\mathcal{I}$ such that $\kappa\in\mathcal{I}_\iota$ and for $k\in\mathcal{I}_\iota$ let n_k be an arbitrary element of $\mathcal{N}_k$. Then $\sum_{k\in\mathcal{I}_\iota}\eta_{n_k}e_{\mathcal{N}_k}=\xi e_{\mathcal{M}_\iota}\in E$. Consequently, by [3, Lemma 4.1], $e_{\mathcal{N}_\kappa}\in E$. Since κ was arbitrarily choosen, each $e_{\mathcal{N}_k}$ is in E and it is a simple matter to show that $(e_{\mathcal{N}_k})_{k\in\mathbb{N}}$ is a Schauder basis of E.

Moreover, $|e_{\mathcal{N}_k}|_{\infty,q}=\max_{j\in\mathcal{N}_k}a_{j,q}$ hence, by [12, Cor. 28.13] and nuclearity, E is isomorphic as a Fréchet space to $\lambda^\infty(\max_{j\in\mathcal{N}_k}a_{j,q})$. The analysis of the proof of [12, Cor. 28.13] shows that this isomorphism is given by $e_{\mathcal{N}_k}\mapsto e_k$ for $k\in\mathbb{N}$, and thus it is also a Fréchet *-algebra isomorphism.

Now, we prove (iii) and (iv). First note that every element of F is the limit of elements of the form $\sum_{k=1}^{M} c_k e_{\mathcal{N}_k}$, where $M \in \mathbb{N}$ and $c_1, \ldots, c_M \in \mathbb{C}$. Therefore, if $\xi \in F$, then $\xi_i = \xi_j$ for $k \in \mathbb{N}$ and $i, j \in \mathcal{N}_k$. This shows that each $\xi \in F$ has the unique series representation $\xi = \sum_{k=1}^{\infty} \xi_{n_k} e_{\mathcal{N}_k}$, where $(n_k)_{k\in\mathbb{N}}$ is an arbitrarily choosen sequence such that $n_k \in \mathcal{N}_k$ for $k \in \mathbb{N}$. Since the series is absolutely convergent, $(e_{\mathcal{N}_k})_{k\in\mathbb{N}}$ is a Schauder basis of F. Statement (iv) follows by the same method as in (ii). □

Corollary 4.4. *Every infinite-dimensional closed *-subalgebra of s is isomorphic as a Fréchet *-algebra to $\lambda^\infty(n_k^q)$ for some strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers. Conversely, if $(n_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers, then $\lambda^\infty(n_k^q)$ is isomorphic as a Fréchet *-algebra to some infinite-dimensional closed *-subalgebra of s. Moreover, every closed *-subalgebra of s is a complemented subspace of s.*

Proof. We apply Proposition 4.3 to the Köthe matrix $(j^q)_{j\in\mathbb{N},q\in\mathbb{N}_0}$. Let $\{\mathcal{N}_k\}_{k\in\mathbb{N}}$ be a family of finite nonempty pairwise disjoint sets of natural numbers. We have

$$\max_{j\in\mathcal{N}_k} j^q = (\max\{j : j \in \mathcal{N}_k\})^q \tag{1}$$

for all $q \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Let $\sigma\colon \mathbb{N} \to \mathbb{N}$ be the bijection for which $(\max\{j : j \in \mathcal{N}_{\sigma(k)}\})_{k\in\mathbb{N}}$ is (strictly) increasing and let $n_k := \max\{j : j \in \mathcal{N}_{\sigma(k)}\}$ for $k \in \mathbb{N}$. Then, by Proposition 4.2,

$$\lambda^\infty\left(\max_{j\in\mathcal{N}_k} j^q\right) \cong \lambda^\infty(n_k^q)$$

as Fréchet *-algebras, and therefore the first two statements follow from Proposition 4.3.

Now, let E be a closed *-subalgebra of s. If E is finite dimensional then, clearly, E is complemented in s. Otherwise, by Proposition 4.3(i), E is a closed linear span of the set $\{e_{\mathcal{N}_k}\}_{k\in\mathbb{N}}$ for some family $\{\mathcal{N}_k\}_{k\in\mathbb{N}}$ of finite nonempty pairwise disjoint sets of natural numbers. Define $\pi\colon s \to E$ by

$$(\pi x)_j := \begin{cases} x_{n_k} & \text{for } j \in \mathcal{N}_{\sigma(k)} \\ 0 & \text{otherwise} \end{cases}$$

where $(n_k)_{k\in\mathbb{N}}$ and σ are as above. From (1) we have for every $q \in \mathbb{N}_0$

$$\begin{aligned} |\pi x|_{\infty,q} &= \sup_{j\in\mathbb{N}} |(\pi x)_j| j^q \le \sup_{k\in\mathbb{N}} |x_{n_k}| \max_{j\in\mathcal{N}_{\sigma(k)}} j^q = \sup_{k\in\mathbb{N}} |x_{n_k}| (\max\{j : j \in \mathcal{N}_k\})^q \\ &= \sup_{k\in\mathbb{N}} |x_{n_k}| n_k^q \le \sup_{j\in\mathbb{N}} |x_j| j^q = |x|_{\infty,q}, \end{aligned}$$

and thus π is well-defined and continuous. Since π is a projection, our proof is complete. □

5. Representations of closed commutative *-subalgebras of $\mathcal{L}(s', s)$ by Köthe algebras

The aim of this section is to describe all closed commutative *-subalgebras of $\mathcal{L}(s', s)$ as Köthe algebras $\lambda^\infty(A)$ for matrices A determined by orthonormal sequences whose elements belong to the space s (Theorem 5.3 and Corollaries 5.4 and 5.5). For the convenience of the reader, we quote two results from [3] (with minor modifications which do not require extra arguments).

For a subset Z of $\mathcal{L}(s', s)$ we will denote by $\mathrm{alg}(Z)$ ($\overline{\mathrm{lin}}(Z)$, resp.) the closed *-subalgebra of $\mathcal{L}(s', s)$ generated by Z (the closed linear span of Z, resp.).

By [3, Lemma 4.4], every closed commutative *-subalgebra E of $\mathcal{L}(s', s)$ admits a special Schauder basis. This basis consists of all nonzero minimal projections in E ([3, Lemma 4.4] shows that these projections are pairwise orthogonal) and we call it the *canonical Schauder basis* of E.

Proposition 5.1. [3, Prop. 4.7] *Every sequence* $\{P_k\}_{k\in\mathcal{N}} \subset \mathcal{L}(s', s)$ *of nonzero pairwise orthogonal projections is the canonical Schauder basis of the algebra* $\mathrm{alg}(\{P_k\}_{k\in\mathcal{N}})$*. In particular,* $\{P_k\}_{k\in\mathcal{N}}$ *is a basic sequence in* $\mathcal{L}(s', s)$*, i.e. it is a Schauder basis of the Fréchet space* $\overline{\mathrm{lin}}(\{P_k\}_{k\in\mathcal{N}})$*.*

Theorem 5.2. [3, Th. 4.8] *Let* E *be a closed commutative infinite-dimensional* **-subalgebra of* $\mathcal{L}(s', s)$ *and let* $\{P_k\}_{k\in\mathbb{N}}$ *be the canonical Schauder basis of* E*. Then*

$$E = \mathrm{alg}(\{P_k\}_{k\in\mathbb{N}}) \cong \lambda^\infty(||P_k||_q)$$

as Fréchet **-algebras and the isomorphism is given by* $P_k \mapsto e_k$ *for* $k \in \mathbb{N}$*.*

Please note that a projection $P \in \mathcal{L}(s', s)$ if and only if it is of the form

$$P\xi = \sum_{k\in I} \langle \xi, f_k\rangle f_k$$

for some finite set I and an orthonormal sequence $(f_k)_{k\in I} \subset s$.

We will also use the identity

$$\lambda^\infty(||\langle\cdot, f_k\rangle f_k||_q) = \lambda^\infty(|f_k|_q) \tag{2}$$

which holds for every orthonormal sequence $(f_k)_{k\in\mathbb{N}} \subset s$. (see [3, Rem. 4.11]).

Now we are ready to state and prove the main result of this section.

Theorem 5.3. *Every closed commutative* **-subalgebra of* $\mathcal{L}(s', s)$ *is isomorphic as a Fréchet* **-algebra to some closed* **-subalgebra of the algebra* $\lambda^\infty(|f_k|_q)$ *for some orthonormal sequence* $(f_k)_{k\in\mathbb{N}} \subset s$*. More precisely, if* E *is an infinite-dimensional closed commutative* **-subalgebra of* $\mathcal{L}(s', s)$ *and* $(\sum_{j\in\mathcal{N}_k}\langle\cdot, f_j\rangle f_j)_{k\in\mathbb{N}}$ *is its canonical Schauder basis for some family of finite pairwise disjoint subsets* $(\mathcal{N}_k)_{k\in\mathbb{N}}$ *of natural numbers and an orthonormal sequence* $(f_j)_{j\in\mathbb{N}} \subset s$*, then* E *is isomorphic as a Fréchet* **-algebra to the closed* **-subalgebra of* $\lambda^\infty(|f_k|_q)$ *generated by* $\{\sum_{j\in\mathcal{N}_k} e_j\}_{k\in\mathbb{N}}$ *and the isomorphism is given by* $\sum_{j\in\mathcal{N}_k}\langle\cdot, f_j\rangle f_j \mapsto \sum_{j\in\mathcal{N}_k} e_j$ *for* $k \in \mathbb{N}$*.*

Conversely, if $(f_k)_{k\in\mathbb{N}} \subset s$ *is an orthonormal sequence, then every closed* **-subalgebra of* $\lambda^\infty(|f_k|_q)$ *is isomorphic as a Fréchet* **-algebra to some closed commutative* **-subalgebra of* $\mathcal{L}(s', s)$*.*

Proof. By Theorem 5.2, $E = \mathrm{alg}\left(\left\{\sum_{j\in\mathcal{N}_k}\langle\cdot, f_j\rangle f_j\right\}_{k\in\mathbb{N}}\right)$ for $(\mathcal{N}_k)_{k\in\mathbb{N}}$ and $(f_j)_{j\in\mathbb{N}} \subset s$ as in the statement. Let F be the closed *-subalgebra of $\lambda^\infty(|f_k|_q)$ generated by $\{\sum_{j\in\mathcal{N}_k} e_j\}_{k\in\mathbb{N}}$. Define

$$\Phi\colon \mathrm{alg}(\{\langle\cdot, f_k\rangle f_k\}_{k\in\mathcal{N}}) \to \lambda^\infty(|f_k|_q)$$

by $\langle\cdot, f_k\rangle f_k \mapsto e_k$, where $\mathcal{N} := \bigcup_{k\in\mathbb{N}}\mathcal{N}_k$. By Proposition 5.1, $\{\langle\cdot, f_k\rangle f_k\}_{k\in\mathcal{N}}$ is the canonical Schauder basis of $\mathrm{alg}(\{\langle\cdot, f_k\rangle f_k\}_{k\in\mathcal{N}})$, and thus Theroem 5.2 and (2) imply that Φ is a Fréchet *-algebra isomorphism. Hence, $(\sum_{j\in\mathcal{N}_k} e_j)_{k\in\mathbb{N}} = (\Phi(\sum_{j\in\mathcal{N}_k}\langle\cdot, f_j\rangle f_j))_{k\in\mathbb{N}}$ is a Schauder basis of $\Phi(E)$ and $\Phi(E)$ is a closed *-subalgebra of $\lambda^\infty(|f_k|_q)$. Therefore,

$$\Phi(E) = \overline{\mathrm{lin}}\left(\left\{\sum_{j\in\mathcal{N}_k} e_j\right\}_{k\in\mathbb{N}}\right) \subset F \subset \Phi(E),$$

whence $\Phi(E) = F$. In consequence $\Phi_{|E}$ is a Fréchet *-algebra isomorphism of E and F, which completes the proof of the first statement.

If now $(f_k)_{k\in\mathbb{N}} \subset s$ is an arbitrary orthonormal sequence then, according to Proposition 5.1, Theorem 5.2 and indentity (2), $\lambda^\infty(|f_k|_q) \cong \mathrm{alg}(\{\langle\cdot, f_k\rangle f_k\}_{k\in\mathbb{N}})$ as Fréchet *-algebras. Consequently, every closed *-subalgebra of $\lambda^\infty(|f_k|_q)$ is isomorphic as a Fréchet *-algebra to some closed *-subalgebra of $\mathrm{alg}(\{\langle\cdot, f_k\rangle f_k\}_{k\in\mathbb{N}})$. □

The following characterization of infinite-dimensional closed commutative *-subalgebras of $\mathcal{L}(s', s)$ is a straightforward consequence of Proposition 4.3 and Theorem 5.3.

Corollary 5.4. *Every infinite-dimensional closed commutative *-subalgebra of $\mathcal{L}(s',s)$ is isomorphic as a Fréchet *-algebra to the algebra $\lambda^\infty(\max_{j\in\mathcal{N}_k}|f_j|_q)$ for some orthonormal sequence $(f_k)_{k\in\mathbb{N}}\subset s$ and some family $\{\mathcal{N}_k\}_{k\in\mathbb{N}}$ of finite nonempty pairwise disjoint sets of natural numbers. In fact, if E is an infinite-dimensional closed commutative *-subalgebra of $\mathcal{L}(s',s)$ and $(\sum_{j\in\mathcal{N}_k}\langle\cdot,f_j\rangle f_j)_{k\in\mathbb{N}}$ is its canonical Schauder basis, then*

$$E\cong\lambda^\infty\left(\max_{j\in\mathcal{N}_k}|f_j|_q\right)$$

as Fréchet *-algebras and the isomorphism is given by $\sum_{j\in\mathcal{N}_k}\langle\cdot,f_j\rangle f_j\mapsto e_k$ for $k\in\mathbb{N}$.

Conversely, if $(f_k)_{k\in\mathbb{N}}\subset s$ is an orthonormal sequence and $\{\mathcal{N}_k\}_{k\in\mathbb{N}}$ is a family of finite nonempty pairwise disjoint sets of natural numbers, then $\lambda^\infty(\max_{j\in\mathcal{N}_k}|f_j|_q)$ is isomorphic as a Fréchet *-algebra to some infinite-dimensional closed commutative *-subalgebra of $\mathcal{L}(s',s)$.

At the end of this section we consider the case of maximal commutative subalgebras of $\mathcal{L}(s',s)$. A closed commutative *-subalgebra of $\mathcal{L}(s',s)$ is said to be *maximal commutative* if it is not properly contained in any larger closed commutative *-subalgebra of $\mathcal{L}(s',s)$.

We say that an orthonormal system $(f_k)_{k\in\mathbb{N}}$ of ℓ_2 is *s-complete*, if every f_k belongs to s and for every $\xi\in s$ the following implication holds: if $\langle\xi,f_k\rangle=0$ for every $k\in\mathbb{N}$, then $\xi=0$. A sequence $\{P_k\}_{k\in\mathbb{N}}$ of nonzero pairwise orthogonal projections belonging to $\mathcal{L}(s',s)$ is called *$\mathcal{L}(s',s)$-complete* if there is no nonzero projection P belonging to $\mathcal{L}(s',s)$ such that $P_kP=0$ for every $k\in\mathbb{N}$.

One can easily show that an orthonormal system $(f_k)_{k\in\mathbb{N}}$ is s-complete if and only if the sequence of projections $(\langle\cdot,f_k\rangle f_k)_{k\in\mathbb{N}}$ is $\mathcal{L}(s',s)$-complete. Hence, by [3, Th. 4.10], closed commutative *-subalgebra E of $\mathcal{L}(s',s)$ is maximal commutative if and only if there is an s-complete sequence $(f_k)_{k\in\mathbb{N}}$ such that $(\langle\cdot,f_k\rangle f_k)_{k\in\mathbb{N}}$ is the canonical Schauder basis of E. Combining this with Corollary 5.4, we obtain the first statement of the following Corollary.

Corollary 5.5. *Every closed maximal commutative *-subalgebra of $\mathcal{L}(s',s)$ is isomorphic as a Fréchet *-algebra to the algebra $\lambda^\infty(|f_k|_q)$ for some s-complete orthonormal sequence $(f_k)_{k\in\mathbb{N}}$. More precisely, if E is a closed maximal commutative *-subalgebra of $\mathcal{L}(s',s)$ with the canonical Schauder basis $(\langle\cdot,f_k\rangle f_k)_{k\in\mathbb{N}}$, then*

$$E\cong\lambda^\infty(|f_k|_q)$$

as Fréchet *-algebras and the isomorphism is given by $\langle\cdot,f_k\rangle f_k\mapsto e_k$ for $k\in\mathbb{N}$.

Conversely, if $(f_k)_{k\in\mathbb{N}}$ is an s-complete orthonormal sequence, then $\lambda^\infty(|f_k|_q)$ is isomorphic as a Fréchet *-algebra to some closed maximal commutative *-subalgebra of $\mathcal{L}(s',s)$.

Proof. In order to prove the second statement, take an arbitrary s-complete orthonormal sequence $(f_k)_{k\in\mathbb{N}}$. By Proposition 5.1 and the remark above our Corollary, $\mathrm{alg}(\{\langle\cdot,f_k\rangle f_k\}_{k\in\mathbb{N}})$ is maximal commutative and from the first statement it follows that it is isomorphic as a Fréchet *-algebra to $\lambda^\infty(|f_k|_q)$. □

It is also worth pointing out the following result.

Proposition 5.6. *Every closed commutative *-subalgebra of $\mathcal{L}(s',s)$ is contained in some maximal commutative *-subalgebra of $\mathcal{L}(s',s)$.*

Proof. Let E be a closed commutative *-subalgebra of $\mathcal{L}(s',s)$. Clearly,

$$\mathcal{X}:=\{\widetilde{E}:\widetilde{E}\text{ commutative }^*\text{-subalgebra of }\mathcal{L}(s',s)\text{ and }E\subset\widetilde{E}\}$$

with the inclusion relation is a partially ordered set. Consider a chain $\mathcal{C}$ in $\mathcal{X}$ and let $E_\mathcal{C}:=\bigcup_{F\in\mathcal{C}}F$. It is easy to check that $E_\mathcal{C}\in\mathcal{X}$, and, of course, $E_\mathcal{C}$ is an upper bound of $\mathcal{C}$. Hence, by the Kuratowski-Zorn lemma, $\mathcal{X}$ has a maximal element; let us call it M. By the continuity of the algebra operations, $\overline{M}^{\mathcal{L}(s',s)}$ is a closed commutative *-subalgebra of $\mathcal{L}(s',s)$, hence from the

maximality of M, we have $M = \overline{M}^{\mathcal{L}(s',s)}$, i.e. M is a (closed) maximal commutative *-subalgebra of $\mathcal{L}(s',s)$ containing E. □

6. Closed commutative *-subalgebras of $\mathcal{L}(s',s)$ with the property (Ω)

In the present section we prove that a closed commutative *-subalgebra of $\mathcal{L}(s',s)$ is isomorphic as a Fréchet *-algebra to some closed *-subalgebra of s if and only if it is isomorphic as a Fréchet space to some complemented subspace of s (Theorem 6.2), i.e. if it has the so-called property (Ω) (see Definition 6.1 below). We also give an example of a closed commutative *-subalgebra of $\mathcal{L}(s',s)$ which is not isomorphic to any closed *-subalgebra of s (Theorem 6.10).

Definition 6.1. A Fréchet space E with a fundamental sequence $(||\cdot||_q)_{q\in\mathbb{N}_0}$ of seminorms has the *property* (Ω) if the following condition holds:

$$\forall p\ \exists q\ \forall r\ \exists \theta\in(0,1)\ \exists C>0\ \forall y\in E' \quad ||y||_q' \leq C||y||_p'^{1-\theta}||y||_r'^{\theta},$$

where E' is the topological dual of E and $||y||_p' := \sup\{|y(x)| : ||x||_p \leq 1\}$.

The property (Ω) (together with the property (DN)) plays a crucial role in the theory of nuclear Fréchet spaces (for details, see [12, Ch. 29]).

Recall that a subspace F of a Fréchet space E is called *complemented* (in E) if there is a continuous projection $\pi\colon E\to E$ with $\operatorname{im}\pi = F$. Since every subspace of $\mathcal{L}(s',s)$ has the property (DN) (and, by [3, Prop. 3.2], the norm $||\cdot||_{\ell_2\to\ell_2}$ is already a dominating norm), [12, Prop. 31.7] implies that a closed *-subalgebra of $\mathcal{L}(s',s)$ is isomorphic to a complemented subspace of s if and only if it has the property (Ω). The class of complemented subspaces of s is still not well-understood (e.g. we do not know whether every such subspace has a Schauder basis – the Pełczyński problem) and, on the other hand, the class of closed *-subalgebras of s has a simple description (see Corollary 4.4). The following theorem implies that, when restricting to the family of closed commutative *-subalgebras of $\mathcal{L}(s',s)$, these two classes of Fréchet spaces coincide.

Theorem 6.2. *Let E be an infinite-dimensional closed commutative *-subalgebra of $\mathcal{L}(s',s)$ and let $(\sum_{j\in\mathcal{N}_k}\langle\cdot,f_j\rangle f_j)_{k\in\mathbb{N}}$ be its canonical Schauder basis. Then the following assertions are equivalent:*

(i) *E is isomorphic as a Fréchet *-algebra to some closed *-subalgebra of s;*
(ii) *E is isomorphic as a Fréchet space to some complemented subspace of s;*
(iii) *E has the property (Ω);*
(iv) $\exists p\ \forall q\ \exists r\ \exists C>0\ \forall k \quad \max_{j\in\mathcal{N}_k}|f_j|_q \leq C\max_{j\in\mathcal{N}_k}|f_j|_p^r$.

In order to prove Theorem 6.2, we will need Lemmas 6.3, 6.4 and Propositions 6.5, 6.7.

The following result is a consequence of nuclearity of closed commutative *-subalgebras of $\mathcal{L}(s',s)$.

Lemma 6.3. *Let $(f_k)_{k\in\mathbb{N}}\subset s$ be an orthonormal sequence and let $(\mathcal{N}_k)_{k\in\mathbb{N}}$ be a family of finite pairwise disjoint subsets of natural numbers. For $r\in\mathbb{N}_0$ let $\sigma_r\colon\mathbb{N}\to\mathbb{N}$ be a bijection such that the sequence $(\max_{j\in\mathcal{N}_{\sigma_r(k)}}|f_j|_r)_{k\in\mathbb{N}}$ is non-decreasing. Then there is $r_0\in\mathbb{N}$ such that*

$$\lim_{k\to\infty}\frac{k}{\max_{j\in\mathcal{N}_{\sigma_r(k)}}|f_j|_r} = 0$$

for all $r\geq r_0$.

Proof. By Corollary 5.4, $\lambda^\infty(\max_{j\in\mathcal{N}_k}|f_j|_q)$ is a nuclear space. Hence, by the Grothendieck-Pietsch theorem (see e.g. [12, Th. 28.15]), for every $q\in\mathbb{N}_0$ there is $r\in\mathbb{N}_0$ such that

$$\sum_{k=1}^\infty \frac{\max_{j\in\mathcal{N}_k}|f_j|_q}{\max_{j\in\mathcal{N}_k}|f_j|_r} < \infty.$$

In particular (for $q = 0$), there is r_0 such that for $r \geq r_0$ we have

$$\sum_{k=1}^{\infty} \frac{1}{\max_{j\in\mathcal{N}_{\sigma_r(k)}} |f_j|_r} = \sum_{k=1}^{\infty} \frac{1}{\max_{j\in\mathcal{N}_k} |f_j|_r} < \infty.$$

Since the sequence $(\max_{j\in\mathcal{N}_{\sigma_r(k)}} |f_j|_r)_{k\in\mathbb{N}}$ is non-decreasing, the conclusion follows from the elementary theory of number series. □

Lemma 6.4. *Let $(a_k)_{k\in\mathbb{N}} \subset [1,\infty)$ be a non-decreasing sequence such that $a_k \geq 2k$ for k big enough. Then there exist a strictly increasing sequence $(b_k)_{k\in\mathbb{N}}$ of natural numbers and $C > 0$ such that*

$$\frac{1}{C} a_k \leq b_k \leq C a_k^2$$

for every $k \in \mathbb{N}$.

Proof. Let $k_0 \in \mathbb{N}$ be such that $a_k \geq 2k$ for $k > k_0$ and choose $C \in \mathbb{N}$ so that

$$\frac{1}{C} a_k \leq k \leq C a_k^2$$

for $k \in \mathcal{N}_0 := \{1, \ldots, k_0\}$. Denote also $\mathcal{N}_1 := \{k \in \mathbb{N} : a_k = a_{k_0+1}\}$ and, recursively, $\mathcal{N}_{j+1} := \{k \in \mathbb{N} : a_k = a_{\max\mathcal{N}_j+1}\}$. Clearly, $\mathcal{N}_j$ are finite, pairwise disjoint, $\bigcup_{j\in\mathbb{N}_0} \mathcal{N}_j = \mathbb{N}$ and $k < l$ for $k \in \mathcal{N}_j$, $l \in \mathcal{N}_{j+1}$.

Let $b_k := k$ for $k \in \mathcal{N}_0$ and let

$$b_{m_j+l-1} := C\lceil\max\{a_{m_j-1}^2, a_{m_j}\}\rceil + l$$

for $j \in \mathbb{N}$ and $1 \leq l \leq |\mathcal{N}_j|$, where $m_j := \min\mathcal{N}_j$ and $\lceil x\rceil := \min\{n \in \mathbb{Z} : n \geq x\}$ stands for the ceiling of $x \in \mathbb{R}$. We will show inductively that $(b_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers such that

$$\frac{1}{C} a_k \leq b_k \leq C a_k^2 \tag{3}$$

for every $k \in \mathbb{N}$.

Clearly, the condition (3) holds for $k \in \mathcal{N}_0$. Assume that $(b_k)_{k\in\mathcal{N}_0\cup\ldots\cup\mathcal{N}_j}$ is a strictly increasing sequence of natural numbers for which the condition (3) holds. For simplicity, denote $m := \min\mathcal{N}_{j+1}$. By the inductive assumption, we obtain $b_{m-1} \leq C a_{m-1}^2$, hence

$$b_m - b_{m-1} \geq C\lceil\max\{a_{m-1}^2, a_m\}\rceil + 1 - C a_{m-1}^2 \geq C a_{m-1}^2 + 1 - C a_{m-1}^2 \geq 1$$

so $b_{m-1} < b_m$, and, clearly, $b_m < b_{m+1} < \ldots < b_{\max\mathcal{N}_{j+1}}$.

Fix $1 \leq l \leq |N_{j+1}|$. We have

$$b_{m+l-1} \geq C a_m = C a_{m+l-1} \geq \frac{1}{C} a_{m+l-1}$$

so the first inequality in (3) holds for $k \in \mathcal{N}_{j+1}$. Next, by assumption, we get

$$a_{m+l-1} \geq 2(m + l - 1), \tag{4}$$

whence

$$l \leq a_{m-l+1} - m + 1. \tag{5}$$

Consider two cases. If $a_m \geq a_{m-1}^2$, then, from (5)

$$\begin{aligned} b_{m-l+1} &= C\lceil a_m\rceil + l = C\lceil a_{m+l-1}\rceil + l \leq 2C a_{m+l-1} + a_{m+l-1} - m + 1 \\ &\leq (2C + 1) a_{m+l-1} \leq C a_{m+l-1}^2, \end{aligned}$$

where the last inequality holds because $C \geq 1$ and, from (4), we have

$$a_{m-l+1} \geq 2(m + l - 1) \geq 2m \geq 2(k_0 + 1) \geq 4.$$

Finally, if $a_{m-1}^2 > a_m$, then, from (4), we obtain (note that, by the definition of $\mathcal{N}_j$ and $\mathcal{N}_{j+1}$, we have $a_{m-1} < a_m$)

$$\begin{aligned} b_{m-l+1} &= C\lceil a_{m-1}^2 \rceil + l \\ &\leq C\lceil (a_m - 1)^2 \rceil + l \\ &= C\lceil a_m^2 - 2a_m + 1 \rceil + l \\ &\leq C(a_m^2 - 2a_m + 2) + l \\ &\leq Ca_m^2 - 2Ca_m + 2C + Cl \\ &= Ca_{m+l-1}^2 - C(2a_{m+l-1} - 2 - l) \\ &\leq Ca_{m+l-1}^2 - C(4(m+l-1) - 2 - l) \\ &= Ca_{m+l-1}^2 - C(4m + 3l - 6) \leq Ca_{m+l-1}^2. \end{aligned}$$

Hence we have shown that the second inequality in (3) holds for $k \in \mathcal{N}_{j+1}$, and the proof is complete. □

Proposition 6.5. *Let E be an infinite-dimensional closed commutative *-subalgebra of $\mathcal{L}(s', s)$ and let $(\sum_{j\in\mathcal{N}_k}\langle\cdot, f_j\rangle f_j)_{k\in\mathbb{N}}$ be its canonical Schauder basis. Moreover, let $(n_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers and let F be the closed *-subalgebra of s generated by $\{e_{n_k}\}_{k\in\mathbb{N}}$. Then the following assertions are equivalent:*

- (i) *E is isomorphic to F as a Fréchet *-algebra;*
- (ii) *$\lambda^\infty(\max_{j\in\mathcal{N}_k}|f_j|_q) \cong \lambda^\infty(n_k^q)$ as Fréchet *-algebras;*
- (iii) *there is a bijection $\sigma\colon \mathbb{N}\to\mathbb{N}$ such that $\lambda^\infty(\max_{j\in\mathcal{N}_{\sigma(k)}}|f_j|_q) = \lambda^\infty(n_k^q)$ as Fréchet *-algebras;*
- (iv) *there is a bijection $\sigma\colon \mathbb{N}\to\mathbb{N}$ such that $\lambda^\infty(\max_{j\in\mathcal{N}_{\sigma(k)}}|f_j|_q) = \lambda^\infty(n_k^q)$ as sets;*
- (v) *there is a bijection $\sigma\colon \mathbb{N}\to\mathbb{N}$ such that*
  - (α) $\forall q\in\mathbb{N}_0\ \exists r\in\mathbb{N}_0\ \exists C>0\ \forall k\in\mathbb{N}\quad \max_{j\in\mathcal{N}_{\sigma(k)}}|f_j|_q \leq Cn_k^r$,
  - (β) $\forall r'\in\mathbb{N}_0\ \exists q'\in\mathbb{N}_0\ \exists C'>0\ \forall k\in\mathbb{N}\quad n_k^{r'} \leq C'\max_{j\in\mathcal{N}_{\sigma(k)}}|f_j|_{q'}$.

Proof. This is an immediate consequence of Proposition 4.2 and Corollary 5.4. □

Remark 6.6. In view of Corollary 4.4, every closed *-subalgebra of s is isomorphic as a Fréchet *-algebra to $\lambda^\infty(n_k^q)$ (i.e. the closed *-subalgebra of s generated by $\{e_{n_k}\}_{k\in\mathbb{N}}$) for some strictly increasing sequence $(n_k)_{k\in\mathbb{N}}\subset\mathbb{N}$, hence Proposition 6.5 characterizes closed commutative *-subalgebras of $\mathcal{L}(s', s)$ which are isomorphic as Fréchet *-algebras to some *-subalgebra of s.

The property (DN) for the space s gives us the following inequality.

Proposition 6.7. *For every $p, r\in\mathbb{N}_0$ there is $q\in\mathbb{N}_0$ such that for all $\xi\in s$ with $||\xi||_{\ell_2} = 1$ the following inequality holds*

$$|\xi|_p^r \leq |\xi|_q.$$

Proof. Take $p, r\in\mathbb{N}_0$ and let $j\in\mathbb{N}_0$ be such that $r\leq 2^j$. Applying iteratively (j-times) the inequality from Proposition 3.2 to $\xi\in s$ with $||\xi||_{\ell_2}=1$ we get

$$|\xi|_p^r \leq |\xi|_p^{2^j} \leq |\xi|_{2^j p},$$

and thus the required inequality holds for $q = 2^j p$. □

Now we are ready to prove Theorem 6.2.

Proof of Theorem 6.2. (i)⇒(ii): By Corollary 4.4, each closed *-subalgebra of s is a complemented subspace of s.

(ii)⇔(iii): See e.g. [12, Prop. 31.7].

(iii)⇒(iv): By Corollary 5.4 and nuclearity (see e.g. [12, Prop. 28.16]),

$$E \cong \lambda^\infty\Big(\max_{j\in\mathcal{N}_k}|f_j|_q\Big) = \lambda^1\Big(\max_{j\in\mathcal{N}_k}|f_j|_q\Big)$$

as Fréchet *-algebras. Hence, by [18, Prop. 5.3], the property (Ω) yields

$$\forall l\ \exists m\ \forall n\ \exists t\ \exists C>0\ \forall k \quad \max_{j\in\mathcal{N}_k}|f_j|_l^t \max_{j\in\mathcal{N}_k}|f_j|_n \le C\max_{j\in\mathcal{N}_k}|f_j|_m^{t+1}.$$

In particular, taking $l=0$, we get (iv).

(iv)⇒(i): Take p from the condition (iv). By Lemma 6.3(ii), there is $p_1\ge p$ and a bijection $\sigma\colon\mathbb{N}\to\mathbb{N}$ such that $(\max_{j\in\mathcal{N}_{\sigma(k)}}|f_j|_{p_1})_{k\in\mathbb{N}}$ is non-decreasing and $\lim_{k\to\infty}\frac{k}{\max_{j\in\mathcal{N}_{\sigma(k)}}|f_j|_{p_1}}=0$. Consequently, for k big enough

$$\max_{j\in\mathcal{N}_{\sigma(k)}}|f_j|_{p_1}\ge 2k,$$

and therefore, by Lemma 6.4, there is a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}\subset\mathbb{N}$ and $C_1>0$ such that

$$\frac{1}{C_1}\max_{j\in\mathcal{N}_{\sigma(k)}}|f_j|_{p_1}\le n_k\le C_1\max_{j\in\mathcal{N}_{\sigma(k)}}|f_j|_{p_1}^2 \tag{6}$$

for every $k\in\mathbb{N}$. Now, by the conditions (iv) and (6), we get that for all q there is r and $C_2:=CC_1^r$ such that

$$\max_{j\in\mathcal{N}_{\sigma(k)}}|f_j|_q\le C\max_{j\in\mathcal{N}_{\sigma(k)}}|f_j|_{p_1}^r\le C_2n_k^r$$

for all $k\in\mathbb{N}$, so the condition (α) from Proposition 6.5(v) holds. Finally, by (6) and Proposition 6.7 we obtain that for all r' there is q' and $C_3:=C_1^{r'}$ such that

$$n_k^{r'}\le C_3\max_{j\in\mathcal{N}_{\sigma(k)}}|f_j|_{p_1}^{2r'}\le C_3\max_{j\in\mathcal{N}_{\sigma(k)}}|f_j|_{q'}$$

for every $k\in\mathbb{N}$. Hence the condition (β) from Proposition 6.5(v) is satisfied, and therefore, by Proposition 6.5, E is isomorphic as a Fréchet *-algebra to the closed *-subalgebra of s generated by $\{e_{n_k}\}_{k\in\mathbb{N}}$. □

Now, we shall give an example of some class of closed commutative *-subalgebras of $\mathcal{L}(s',s)$ which are isomorphic to closed *-subalgebras of s.

Example 6.8. Let $\mathbb{H}_1:=[1]$. We define recursively *Hadamard matrices*

$$\mathbb{H}_{2^n}:=\begin{bmatrix}\mathbb{H}_{2^{n-1}} & \mathbb{H}_{2^{n-1}}\\ \mathbb{H}_{2^{n-1}} & -\mathbb{H}_{2^{n-1}}\end{bmatrix}$$

for $n\in\mathbb{N}$. Then the matrices $\widehat{\mathbb{H}}_{2^n}:=2^{-\frac{n}{2}}\mathbb{H}_{2^n}$ are unitary, and thus their rows form an orthonormal system of 2^n vectors. Now fix an arbitrary sequence $(d_n)_{n\in\mathbb{N}}\subset\mathbb{N}_0$ and define

$$U:=\begin{bmatrix}\widehat{\mathbb{H}}_{2^{d_1}} & 0 & 0 & \dots\\ 0 & \widehat{\mathbb{H}}_{2^{d_2}} & 0 & \dots\\ 0 & 0 & \widehat{\mathbb{H}}_{2^{d_3}} & \\ \vdots & \vdots & & \ddots\end{bmatrix}.$$

Let f_k denote the k-th row of the matrix U. Then $(f_k)_{k\in\mathbb{N}}$ is an orthonormal basis of ℓ_2 and clearly each f_k belongs to s. We will show that the closed (maximal) commutative *-subalgebra $\mathrm{alg}(\{\langle,\cdot,f_k\rangle f_k\}_{k\in\mathbb{N}})$ of $\mathcal{L}(s',s)$ is isomorphic to some closed *-subalgebra of s. By Theorem 6.2, it is enough to prove that

$$\exists p\ \forall q\ \exists r\ \exists C>0\ \forall k \quad |f_k|_{\infty,q}\le C|f_k|_{\infty,p}^r. \tag{7}$$

Fix $q \in \mathbb{N}_0$, $k \in \mathbb{N}$ and find $n \in \mathbb{N}$ such that $2^{d_1} + \ldots + 2^{d_{n-1}} < k \leq 2^{d_1} + \ldots + 2^{d_n}$. Then

$$\frac{|f_k|_{\infty,q}}{|f_k|_{\infty,1}^{2q}} = \frac{2^{-\frac{d_n}{2}}(2^{d_1} + \ldots + 2^{d_n})^q}{2^{-d_n q}(2^{d_1} + \ldots + 2^{d_n})^{2q}} = 2^{d_n(q-1/2)}(2^{d_1} + \ldots + 2^{d_n})^{-q} \leq 1$$

and thus the condition (7) holds with $p = C = 1$ and $r = 2q$.

The next theorem solves in negative [3, Open Problem 4.13]. In contrast to the algebra s, whose all closed *-subalgebras are complemented subspaces of s (Corollary 4.4), Theorems 6.2 and 6.10 imply that there is a closed commutative *-subalgebra of $\mathcal{L}(s', s)$ which is not complemented in $\mathcal{L}(s', s)$ (otherewise it would have the property (Ω), see [12, Prop. 31.7]). In the proof we will use the following identity.

Lemma 6.9. *For every increasing sequence $(\alpha_j)_{j\in\mathbb{N}} \subset (0, \infty)$ and every $p \in \mathbb{N}$ we have*

$$\sup_{j\in\mathbb{N}} \left(\alpha_j^{p-j+1} \cdot \prod_{i=1}^{j-1} \alpha_i\right) = \prod_{i=1}^{p} \alpha_i.$$

Proof. For $j \geq p + 1$ we get

$$\frac{\alpha_j^{p-j+1} \cdot \prod_{i=1}^{j-1} \alpha_i}{\prod_{i=1}^{p} \alpha_i} = \alpha_j^{p-j+1} \cdot \prod_{i=p+1}^{j-1} \alpha_i = \frac{\prod_{i=p+1}^{j-1} \alpha_i}{\alpha_j^{j-p-1}} \leq 1$$

and, similarly, for $j \leq p - 1$ we obtain

$$\frac{\alpha_j^{p-j+1} \cdot \prod_{i=1}^{j-1} \alpha_i}{\prod_{i=1}^{p} \alpha_i} = \frac{\alpha_j^{p-j+1}}{\prod_{i=j}^{p} \alpha_i} \leq 1.$$

Since $\alpha_p^{p-p+1} \cdot \prod_{i=1}^{p-1} \alpha_i = \prod_{i=1}^{p} \alpha_i$, the supremum is attained for $j = p$, and we are done. □

Theorem 6.10. *There is a closed commutative *-subalgebra of $\mathcal{L}(s', s)$ which is not isomorphic to any closed *-subalgebra of s.*

Proof. Let m_k be the k-th prime number, $N_{k,1} := m_k$, $N_{k,j+1} := m_k^{N_{k,j}}$ for $j, k \in \mathbb{N}$. Denote $a_{k,1} := c_k$ and

$$a_{k,j} := c_k \frac{\prod_{i=1}^{j-1} N_{k,i}}{N_{k,j}^{j-1}}$$

for $j \geq 2$, where the sequence $(c_k)_{k\in\mathbb{N}}$ is choosen so that $||(a_{k,j})_{j\in\mathbb{N}}||_{\ell_2} = 1$, i.e.

$$c_k := \left(\sum_{j=1}^{\infty} \left(\frac{\prod_{i=1}^{j-1} N_{k,i}}{N_{k,j}^{j-1}}\right)^2\right)^{-1/2}.$$

The numbers c_k are well-defined, because, by Lemma 6.9,

$$\begin{aligned}\sum_{j=1}^{\infty} \left(\frac{\prod_{i=1}^{j-1} N_{k,i}}{N_{k,j}^{j-1}}\right)^2 &= \sum_{j=1}^{\infty} \left(N_{k,j}^{-j+1} \cdot \prod_{i=1}^{j-1} N_{k,i}\right)^2 = \sum_{j=1}^{\infty} \frac{1}{N_{k,j}^2} \left(N_{k,j}^{1-j+1} \cdot \prod_{i=1}^{j-1} N_{k,i}\right)^2 \\ &\leq \sup_{j\in\mathbb{N}} \left(N_{k,j}^{1-j+1} \cdot \prod_{i=1}^{j-1} N_{k,i}\right)^2 \sum_{j=1}^{\infty} \frac{1}{N_{k,j}^2} = N_{k,1}^2 \sum_{j=1}^{\infty} \frac{1}{N_{k,j}^2} < N_{k,1}^2 \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty.\end{aligned}$$

Finally, define an orthonormal sequence $(f_k)_{k\in\mathbb{N}}$ by

$$f_k := \sum_{j=1}^{\infty} a_{k,j} e_{N_{k,j}}.$$

We will show that $\operatorname{alg}(\{\langle\cdot, f_k\rangle f_k\}_{k\in\mathbb{N}})$ is a closed *-subalgebra of $\mathcal{L}(s', s)$ which is not isomorphic as an algebra to any closed *-subalgebra of s. By Theorem 6.2 and nuclearity, it is enough to show that each f_k belongs to s and for every $p, r \in \mathbb{N}$ the following condition holds

$$\lim_{k\to\infty} \frac{|f_k|_{\infty,p+1}}{|f_k|^r_{\infty,p}} = \infty,$$

where $|\xi|_{\infty,q} := \sup_{j\in\mathbb{N}} |\xi_j| j^q$.

Note first that $|f_k|_{\infty,p} = a_{k,p}N^p_{k,p}$. In fact, by Lemma 6.9, we get

$$|f_k|_{\infty,p} = \sup_{j\in\mathbb{N}} a_{k,j}N^p_{k,j} = c_k \sup_{j\in\mathbb{N}} \left(N^p_{k,j} \cdot \frac{\prod_{i=1}^{j-1} N_{k,i}}{N^{j-1}_{k,j}}\right) = c_k \sup_{j\in\mathbb{N}} \left(N^{p-j+1}_{k,j} \cdot \prod_{i=1}^{j-1} N_{k,i}\right)$$
$$= c_k \prod_{i=1}^{p} N_{k,i} = c_k N^p_{k,p} \cdot \frac{\prod_{i=1}^{p-1} N_{k,i}}{N^{p-1}_{k,p}} = a_{k,p}N^p_{k,p}.$$

In particular, $f_k \in s$ for $k \in \mathbb{N}$. Next, for $j, k \in \mathbb{N}$, we have

$$\frac{a_{k,j+1}N^j_{k,j+1}}{a_{k,j}} = \frac{c_k N^j_{k,j+1} \cdot \frac{\prod_{i=1}^{j} N_{k,i}}{N^j_{k,j+1}}}{c_k \frac{\prod_{i=1}^{j-1} N_{k,i}}{N^{j-1}_{k,j}}} = \frac{\prod_{i=1}^{j} N_{k,i}}{\frac{\prod_{i=1}^{j-1} N_{k,i}}{N^{j-1}_{k,j}}} = N^j_{k,j}.$$

Moreover, for every $j, r \in \mathbb{N}$ we get

$$\frac{N_{k,j+1}}{N^r_{k,j}} = \frac{m_k^{N_{k,j}}}{N^r_{k,j}} \geq \frac{2^{N_{k,j}}}{N^r_{k,j}} \xrightarrow[k\to\infty]{} \infty,$$

and clearly $a_{k,j} \leq 1$ for $j, k \in \mathbb{N}$. Hence, for $p, r \in \mathbb{N}$ we obtain

$$\frac{|f_k|_{\infty,p+1}}{|f_k|^r_{\infty,p}} = \frac{a_{k,p+1}N^{p+1}_{k,p+1}}{a^r_{k,p}N^{pr}_{k,p}} = \frac{a_{k,p+1}N^p_{k,p+1}}{a_{k,p}} \cdot \frac{1}{a^{r-1}_{k,p}} \cdot \frac{N_{k,p+1}}{N^{pr}_{k,p}} = N^p_{k,p} \cdot \frac{1}{a^{r-1}_{k,p}} \cdot \frac{N_{k,p+1}}{N^{pr}_{k,p}}$$
$$\geq \frac{N_{k,p+1}}{N^{pr}_{k,p}} \xrightarrow[k\to\infty]{} \infty,$$

which is the desired conclusion. □

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